Covariance Maximized Lambda 4: An Introduction of a Low-Biased Reliability Coefficient

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Abstract

Combining the strengths of coefficient alpha and maximized reliability coefficients, this paper introduces an estimator based on a maximization method that utilizes a covariance matrix to obtain a subset of split-half reliabilities. The mean of the vector of split-half reliabilities can then be used to estimate reliability. Only the maximum of this vector (Guttman's maximal λ_4) has been explored previously despite evidence that it may overestimate reliability in small samples. Computation theory, algorithm and code based in R are provided for computation for the estimator. Simulations were used to assess the performance of the estimator in a variety of data structures while also allowing direct comparisons to the greatest lower bound, Mcdonald's omega, and coefficient alpha.

Keywords: *Split-half, reliability, glb, omega, alpha*

Introduction

Estimating reliability has taken many forms since Spearman's conceptualization of errors in observation (Spearman, 1904). Despite the long history, the field has struggled to find an estimator that is appropriate in realistic settings. Although Cronbach's (1951) piece on coefficient alpha is one of the most heavily cited contributions to psychometric theory, the statistic is often misused (Sijtsma, 2009a). Cronbach (1951) was able to prove that coefficient alpha is the mean of all split-half reliabilities by making the stringent assumptions that each of the items in the scale exhibit tau-equivalence (items have equivalent factor loadings) and are unidimensional (Novick and Lewis, 1967). When a scale meets these assumptions and the data is at a population level of analysis, all split-half reliabilities will be equivalent. So while one could attain the appropriate parameter by selecting the maximum or minimum split-half reliability the mean is an unbiased estimator in a sample. Although valid, the assumptions made in the Cronbach's (1951) proof are often not met in realistic settings.

Guttman (1945) favored his λ_4 reliability coefficient, a split-half reliability estimate, because it is does not assume tau-equivalence or unidimensionality. Guttman recommended splitting the test items in a way that maximizes this estimate but did not provide a specific method for doing so. Nevertheless, multiple researchers have made developments in this regard over the last 60 years. The greatest lower bound (ρ_+), originally proposed by Bentler (1972), posited that the largest reliability attainable is found by minimizing the trace of the covariance matrix (the variances) while keeping the matrix

positive semi-definite (all eigenvalues are non-negative). Maximized λ_4 was developed by Callender and Osburn (1977) and utilized an algebraic algorithm for assigning items into halves with an equivalent number of items. Mcdonald's omega (ω_t), an estimate of the total reliability of the scale, is similar to Guttman's (1945) λ_6 and uses item's uniquenesses to estimate the error variance of the scale (Revelle, 2012).

Proper use of these reliability estimates has been a source of confusion and contention. Sijtsma (2009a) recommended using (ρ_+) despite the estimate's large positive bias in small and moderately large samples (ten Berge and Socan, 2004) and it's tendency to capitalizes on chance (Cronbach, 1988). Instead, Revelle and Zinbarg (2009) suggested use of McDonald's (1999) ω_t and presented evidence that estimates from ω_t were higher than (ρ_+). Although ω_t has not been studied explicitly in terms of bias, Revelle's (2009) results suggest that it also has a large postive bias. Also of note, Osburn (2000) provided evidence that his maximized λ_4 was consistently accurate in parallel form, tau equivalent, congeneric, and different levels of heterogeneous data factor solutions. However, this simulation was performed on population covariance matrices and not with sampled data.

The ubiquity of Cronbach's alpha (1951) and the numerous calls for use of "better" reliability estimators (Revelle and Zinbarg, 2009; Sijtsma, 2009a; Osburn, 2000; Callender and Osburn, 1977, 1979; Jackson and Agunwamba, 1977; Woodhouse and Jackson, 1977) provide indication that psychometricians and researchers need a proper maximized reliability estimate. The most recently developed techniques (e.g. McDonald (1978); Jackson and Agunwamba (1977); Osburn (2000)) for calculating reliability exhibit bias

and will be ignored until a bias corrected method is proposed and validated (Yaun and Bentler, 2002; ten Berge and Socan, 2004; Sijtsma, 2009a,b; Revelle and Zinbarg, 2009). Fortunately, this is an attainable goal.

By combining the logic of coefficient α and maximized reliability coefficients the strengths of both techniques may be attained. Generally speaking a maximization procedure could yield a subset of maximized split-half reliabilities. One could then take the mean of the split-half reliabilities to minimize capitalization on error. The major purpose of this paper is to introduce the method behind the computation of a statistic and provide evidence that it is based on minimal assumptions and exhibits a low level of bias.

The layout of the paper will proceed as follows. The computational theory of the estimator is described as well as instructions for implementation. The subsequent section includes empirical results observed from a simulation study. The paper is concluded with a discussion and code for use in R (R Development Core Team, 2011) for direct calculation in Appendix A.

Computational Theory

Osburn (2000) suggested using an estimation procedure for calculating maximized λ_4. Specifically, he advised finding the largest covariance and splitting those two items onto different halves (paired-item separations) and then iterating this procedure, ignoring previously split items, until all of the items have been placed on one of the splits. Although, Osburn only discussed calculating λ_4 for a single split, it can actually be calculated on every possible split that maintains the paired-item separations. This is the maximization

procedure provides a method for obtaining a subset of maximized λ_4s. For the direct calculation of λ_4 Equation 1 can be used. We will let Σ be the $p * p$ covariance matrix for the analysis. Let t be a vector of length p with elements 1 or -1 that indicate the split each item is placed within and let *1* be the unit vector of length p.

$$\lambda_4 = 1 - \left(\frac{t'\Sigma t}{1'\Sigma 1}\right) \tag{1}$$

By replicating this calculation on a single set of items several times a vector of λ_4s can be obtained that also maintain the inital paired-item separations (compare Table 1 to Table 2). It is evident that these splits will provide different values in a sample while also fitting to the rules of maximization defined above.

Table 1: Original Split

Half 1	Half 2
X_1	Y_1
X_2	Y_2
X_3	Y_3
X_4	Y_4

Table 2: Equivalent Split

Half 1	Half 2
X_1	Y_1
Y_2	X_2
X_3	Y_3
Y_4	X_4

Finally, Equation 1 is performed on every t vector that maintains the inter-item separations. Once a vector τ is made of all the possible λ_4 estimates, the mean or median of the vector can be used as the final estimate of reliability. The maximum of this vector would be the same as maximum λ_4.

$$\lambda_{4(C)} = mean(\hat{\lambda}_{4_{1:n}}, \hat{\lambda}_{4_{2:n}}, \hat{\lambda}_{4_{3:n}} ... \hat{\lambda}_{4_{n:n}}) = mean(\tau) \tag{2}$$

The procedure for computing Covariance Maximized Lambda 4 ($\lambda_{4(C)}$) is

described below.

1. Find the largest covariance between two items and place those items on separate splits.

2. Reiterate 1 while ignoring previously split items, thus creating a vector t of -1, 1 elements.

3. Generate every combination of -1 and 1 elements for t that maintains the paired-item separations.

4. Compute $\hat{\lambda}_4$ for each t_i using Equation 1 creating a vector τ.

5. Take the mean or median of τ for $\lambda_{4(C)}$. If one desires Guttman's $\lambda_{4(max)}$ take the max of τ.

Simulation Method

To establish this new method for estimating reliability the method was tested on simulated data and compared to other estimators. The simulations were performed in R system for statistical computing (R Development Core Team, 2011). Three factor structures were simulated, parallel, tau-equivalent, and congeneric, with either 1, 3, or 5 factors and 5 different sample sizes (50, 100, 400, 1000, and 2000). Each simulation included 500 samples yielding 500 estimates of the greatest lower bound (ρ_+), McDonald's Omega (ω_t) (1999), coefficient alpha (α) and covariance maximized lambda 4 ($\lambda_{4(C)}$). McDonald's ω_t was calculated using the **factanal** function from **stats** package in

R which uses maximum likelihood. The equations for the estimators are described below.

$$\rho_{+} = 1 - \frac{tr(\Sigma^{*})}{1'\Sigma 1} = \frac{1'\Sigma^{*}1}{1'\Sigma 1} \tag{3}$$

Where Σ^{*} is equivalent to Σ but the trace (tr) has been minimized.

$$\omega_t = 1 - \left(\frac{\sum u_i^2}{1'\Sigma 1} \right) \tag{4}$$

Because ω_t requires specification of the number of factors, the number of eigenvalues greater than one was utilized to make that determination.

$$\alpha = \frac{1}{n-1} \left(\frac{tr\Sigma}{1'\Sigma 1} \right) \tag{5}$$

The population covariance structures were established using a confirmatory factor model $\Sigma = \Lambda\Phi\Lambda' + \Psi$, where Ψ is a diagonal $p * p$ matrix of error variances, Λ is a $p * n$ matrix of the measurement model and Φ is an $n * n$ matrix of the correlations between the factors, where n is the number of factors. For simulation, the population covariance matrix was used in the **mvrnorm** function from the **MASS** package in R. Population internal consistency reliabilities are defined in Equation 6.

$$\rho = \frac{1'\Lambda\Phi\Lambda' 1}{1'\Sigma 1} = 1 - \frac{1'\Psi 1}{1'\Sigma 1} \tag{6}$$

Models with more that 1 factor required specification of the correlations between the factors via the Φ matrix from the confirmatory model. The Φ matrices for the models were set up with 1s on the diagonal and .3s on the off

diagonal specifying a correlation between all latent variables of .3. Shortened examples of the confirmatory models for each factor structure are provided below. For the multidimensional models a Φ matrix as described above was included and the terms of this example were extended to account for the increase in items and factors.

Parallel Models. The parallel models were designed with equal factor loadings of .6 for each item and equal error variances of $.6^2$.

$$\Sigma_1 = \begin{pmatrix} .6 \\ .6 \\ .6 \\ .6 \end{pmatrix} * \begin{pmatrix} .6 & .6 & .6 & .6 \end{pmatrix} + \begin{pmatrix} .6^2 & 0 & 0 & 0 \\ 0 & .6^2 & 0 & 0 \\ 0 & 0 & .6^2 & 0 \\ 0 & 0 & 0 & .6^2 \end{pmatrix} \tag{7}$$

Tau-Equivalent Models. The tau-equivalent models also have equated factor loadings of .6 but have error variances of $.6^2$, $.7^2$, $.8^2$, and $.9^2$.

$$\Sigma_2 = \begin{pmatrix} .6 \\ .6 \\ .6 \\ .6 \end{pmatrix} * \begin{pmatrix} .6 & .6 & .6 & .6 \end{pmatrix} + \begin{pmatrix} .6^2 & 0 & 0 & 0 \\ 0 & .7^2 & 0 & 0 \\ 0 & 0 & .8^2 & 0 \\ 0 & 0 & 0 & .9^2 \end{pmatrix} \tag{8}$$

Congeneric. The congeneric models have different factor loadings for each item of .5, .6, .7, and .8 and also have error variances of $.6^2$, $.7^2$, $.8^2$, and $.9^2$.

$$\Sigma_3 = \begin{pmatrix} .5 \\ .6 \\ .7 \\ .8 \end{pmatrix} * \begin{pmatrix} .5 & .6 & .7 & .8 \end{pmatrix} + \begin{pmatrix} .6^2 & 0 & 0 & 0 \\ 0 & .7^2 & 0 & 0 \\ 0 & 0 & .8^2 & 0 \\ 0 & 0 & 0 & .9^2 \end{pmatrix} \tag{9}$$

These models were chosen for proper comparisons to existing estimators known strengths.

The expected values of each statistic were tested for biasedness and consistentency. Bias was calculated by Equation 10 where $\hat{\rho}$ is a general term for the expected value of the reliability statistics.

$$\theta = \rho - \hat{\rho} \tag{10}$$

Consistency S was computed by taking the standard deviation of the sample estimates $\hat{\rho^*}$ of each statistic.

$$Consistency = sd(\hat{\rho^*}) \tag{11}$$

The *MSE*, the second moment of the error, was computed with Equation 12 and provides a way to compare statistics that differ on biasedness and consistency.

$$MSE = \sigma^2 + \theta^2 \tag{12}$$

The results from the simulations are described in the following section.

Results

Parallel Models (Table 3). As expected α performs well with a very low amount of bias in the unidimensional factor structure but has a large negative bias in the multidimensional models. Also as expected, ρ_+ exhibits an unacceptable level of bias for each model in sample sizes less than 1000. The bias of ρ_+ seems to increase as the number of factors/items increases. Despite the large amount of bias, ρ_+'s estimates are consistent. Although $\lambda_{4(C)}$ utilizes an unneccessary maximization step for the unidimensional case, it exhibits a low level of bias and has either lower or equivalent consistency as α. In the multidimensional cases $\lambda_{4(C)}$ has the lowest level of bias and MSE. Also performing very well, ω_t does not have a bias greater than .01 in any of the factor structures. In summation, ω_t and $\lambda_{4(C)}$ perform best and have the lowest MSE in all of the models.

Tau-Equivalent Models (Table 4). The results from these models very closely resemble the results from the parallel models. First, α performs well in the unidimensional case but exhibits a large negative bias in the multidimensional cases. Second, ρ_+ exhibits a large positive bias that is as high as .10 in the 5 factor model. Third, $\lambda_{4(C)}$ perfoms very well and has the lowest level of bias and MSE in the multidimensional models. Lastly, ω_t exhibits a bias in sample sizes of 100 and smaller but has low consistency and MSE across the three models.

Congeneric Models (Table 5). Although α performed reasonably well in the parallel and tau-equivalent unidimensional models, ω_t and $\lambda_{4(C)}$ have an equivalent level or smaller bias, consistency, and MSE. Similar to the pre-

vious models, ρ_+ exhibits an unacceptable positive bias in samples smaller than 1000.

Discussion

Recent changes have impacted the design and development of psychometric tests. It seems that acheiving a test that satisfies assumptions of parallel forms requires asking the same question 10 different ways. This methodology is ineffective, repetitive for test respondents, and the resulting test may have low predictive qualities. Different analytical techniques that explore factor structures with a general factor in addition to specific factors with a subset of the items both in an exploratory sense (Jennrich and Bentler, 2011, 2012) and a confirmatory sense (Holzinger and Swineford, 1937) are becoming more commonly utilized. Because the field is moving in this direction a set of tools that offers more flexibility is needed. Specifically, there is a need for an internal consistency reliability estimator for multidimensional factor structures that does not exhibit a large bias. A theory building on the ideas of Guttman and Osburn has been provided that establishes a modern perspective on reliability estimation that corrects the issues of previous estimators. A computational methodology has been provided along with code in R for direct computation. As expected ω_t and $\lambda_{4(C)}$ outperformed ρ_+ and α in general. Nevertheless, α performed well in the unidimensional case, as expected (Novick and Lewis, 1967). If one was attenuating correlations it would be advantageous to utilize either ω_t or $\lambda_{4(C)}$ as to not overattenuate. In comparing ω_t and $\lambda_{4(C)}$ across the different factor structures it is noticable

that they provide similar results. Nevertheless, $\lambda_{4(C)}$ does not require prior knowledge of the number of latent factors and demonstrates a lower level of bias as the number of factors increases or when the sample size is small. The bias of ρ_+ found by previous researchers (Yaun and Bentler, 2002; ten Berge and Socan, 2004; Sijtsma, 2009a,b; Revelle and Zinbarg, 2009; Li and Bentler, 2011) was confirmed and increases as the number of items and/or factors increase. $\lambda_{4(C)}$ also yields a marked improvement over ρ_+ with a smaller level of bias in all three populations. The computational methodology relies heavily on a method described in Osburn (2000). The expansion includes the generation of all possible split half relibilities that still maintain inter-item separations from the maximization procedure. With a moderate to large number of items calculation can be computationally extensive but comparable to modern bootstrapping procedures. On a 2.66 GHz Intel Core 2 Duo MacBook Pro a dataset with 10,000 observations and 30 variables computation time varied from .653 to .942 seconds. Further code developments such as vectorization or use of different languages that handle computation more quickly (e.g. Fortran or C) are also possibilities. It would also be useful to compare results provided by taking the median of τ instead of the mean. By using the median split it may be possible to use bootstrapping with a specific $\lambda_{4(C)}$ to estimate confidence intervals while also cutting down on computation. Issues such as this are will be the focus of my ongoing research.

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Appendix A

```
cov.lambda4<-function (x, show.lambda4s = FALSE, show.splits = TRUE)
{
    nvar <- dim(x)[2]
    n <- dim(x)[1]
    p <- dim(x)[2]
    if (n == p)
     sigma <- as.matrix(x)
    else sigma <- var(x, use = "pairwise")
    sigma.split <- as.data.frame(sigma)
    sigma.split[upper.tri(sigma.split, diag = TRUE)] <- 0
    sigma.split2 <- sigma - diag(sigma)
    xy <- matrix(ncol = 2, nrow = nvar/2)
    for (o in 1:(nvar/2)) {
     x.m <- which(sigma.split == max(
      sigma.split), arr.ind = TRUE)[1,]
       xy[o, 1] <- x.m[1]
       xy[o, 2] <- x.m[2]
       sigma.split[(x.m[1]), ] <- -999999
       sigma.split[(x.m[1])] <- -999999
       sigma.split[(x.m[2])] <- -999999
       sigma.split[(x.m[2]), ] <- -999999
    }
    Ahalf <- xy[, 1]
    Bhalf <- xy[, 2]
    items.seq <- seq(1:nvar)
    lst <- c(Ahalf, Bhalf)
    lftout <- which(items.seq %in% lst == FALSE)
    if (length(c(Ahalf, Bhalf)) != length(items.seq)) {
     Bhalf <- c(Bhalf, lftout)
    }
    Ani <- length(Ahalf)
    Bni <- length(Bhalf)
    Acombs <- bin.combs(Ani)
    lencombs <- nrow(Acombs)
    t1t.temp <- (as.numeric(items.seq %in% Ahalf) - 0.5) * 2
    t1t.splits <- t(matrix(data = rep(t1t.temp, lencombs),
```

```
     nrow = nvar, ncol = lencombs))
    full <- cbind(Acombs, Acombs)
    if (Ani != Bni) {
     full <- cbind(full, rep(1, lencombs))
    }
    full[, c(Ahalf, Bhalf)] <- full[, seq(1:nvar)]
    if (Ani != Bni) {
     covt <- which(sigma.split2[lftout, ] == max(
sigma.split2[lftout, ]))
    }
    if (Ani != Bni) {
     full[, lftout] <- -t1t.temp[covt]
    }
    t1t.matrix <- (full * t1t.splits)/2 + 0.5
    t2.matrix <- (t(t1t.matrix) - 1) * -1
    onerow <- rep(1, ncol(t1t.matrix))
    onerow <- t(onerow)
    onevector <- t(onerow)
    l4.vect <- rep(NA, lencombs)
    for (i in 1:lencombs) {
     l4.vect[i] <- (4 * (t1t.matrix[i, ] %*% sigma %*%
      t2.matrix[, i]))/(onerow %*% sigma) %*% onevector
    }
    if (show.splits == TRUE) {
     sl4 <- sort(l4.vect)
      Min.Split <- t1t.matrix[which(l4.vect == sl4[1]), ]
      Median.Split <- t1t.matrix[which(l4.vect ==
 sl4[round(lencombs/2)]), ]
      Max.Split <- t1t.matrix[which(l4.vect ==
 sl4[lencombs]), ]
      Splits <- data.frame(Min.Split, Median.Split, Max.Split)
    }
    Max <- max(l4.vect)
    Mean <- mean(l4.vect)
    Median <- median(l4.vect)
    Minimum <- min(l4.vect)
    lambda4s <- lencombs
    Items <- nvar
    lambda4 <- data.frame(Mean, Max, Median, Minimum)
    Analysis.Details <- data.frame(Items, lambda4s)
    if (show.lambda4s == FALSE) {
```

```
    if (show.splits == TRUE) {
     result <- list(lambda4 = lambda4,
Analysis.Details = Analysis.Details,
      Splits = Splits)
    }
    else {
     result <- list(lambda4 = lambda4,
Analysis.Details = Analysis.Details)
    }
   }
   if (show.lambda4s == TRUE) {
    if (show.splits == TRUE) {
     result <- list(lambda4 = lambda4,
Analysis.Details = Analysis.Details,
      lambda4s = l4.vect, Splits = Splits)
    }
    else {
     result <- list(lambda4 = lambda4,
Analysis.Details = Analysis.Details,
      lambda4s = l4.vect)
    }
   }
return(result)
}
```

Table 3: Parallel Factor Structure

		1 Factor				3 Factor				5 Factor			
	N	α	ρ_+	λ_4	ω_t	α	ρ_+	λ_4	ω_t	α	ρ_+	λ_4	ω_t
Bias	50	.00	.04	.01	.00	-.07	.06	.00	.01	-.07	.07	.00	.01
	100	.00	.03	.01	.00	-.07	.05	.00	.00	-.06	.05	.00	.01
	400	.00	.01	.00	.00	-.07	.02	.00	.00	-.06	.03	.00	.00
	1000	.00	.01	.00	.00	-.07	.01	.00	.00	-.06	.02	.00	.00
	2000	.00	.01	.00	.00	-.07	.01	.00	.00	-.06	.01	.00	.00
Consistency	50	.03	.02	.02	.02	.05	.02	.03	.03	.04	.01	.02	.02
	100	.02	.01	.02	.02	.03	.01	.02	.02	.02	.01	.01	.01
	400	.01	.01	.01	.01	.02	.01	.01	.01	.01	.01	.01	.01
	1000	.01	.01	.01	.01	.01	.01	.01	.01	.01	.00	.00	.00
	2000	.00	.00	.00	.00	.01	.00	.00	.00	.01	.00	.00	.00
MSE * 100	50	.07	.20	.06	.05	.77	.45	.10	.08	.56	.47	.05	.05
	100	.03	.11	.03	.03	.63	.23	.04	.04	.45	.25	.02	.02
	400	.01	.03	.01	.01	.52	.06	.01	.01	.38	.07	.01	.01
	1000	.00	.01	.00	.00	.49	.02	.00	.00	.37	.03	.00	.00
	2000	.00	.01	.00	.00	.49	.01	.00	.00	.37	.01	.00	.00

Table 4: Tau-Equivalent Factor Structure

		1 Factor				3 Factor				5 Factor			
	N	α	ρ_+	λ_4	ω_t	α	ρ_+	λ_4	ω_t	α	ρ_+	λ_4	ω_t
Bias	50	-.01	.06	.02	.02	-.08	.09	.00	.03	-.06	.10	.01	.04
	100	.00	.04	.01	.01	-.07	.07	.00	.02	-.06	.07	.00	.02
	400	.00	.02	.01	.01	-.06	.03	.00	.01	-.06	.04	.00	.01
	1000	.00	.01	.00	.01	-.06	.02	.00	.01	-.06	.02	.00	.01
	2000	.00	.01	.00	.01	-.06	.02	.00	.01	-.06	.02	.00	.01
Consistency	50	.04	.02	.03	.03	.06	.03	.05	.03	.05	.01	.03	.02
	100	.03	.02	.02	.02	.04	.02	.03	.03	.03	.01	.02	.02
	400	.01	.01	.01	.01	.02	.01	.01	.01	.02	.01	.01	.01
	1000	.01	.01	.01	.01	.01	.01	.01	.01	.01	.01	.01	.01
	2000	.01	.01	.01	.01	.01	.01	.01	.01	.01	.00	.01	.01
MSE * 100	50	.14	.45	.14	.10	.97	.94	.20	.22	.63	1.07	.11	.23
	100	.07	.23	.07	.05	.64	.48	.10	.10	.48	.56	.05	.07
	400	.02	.07	.02	.02	.46	.14	.02	.03	.35	.15	.01	.02
	1000	.01	.03	.01	.01	.43	.05	.01	.01	.34	.06	.01	.01
	2000	.00	.01	.00	.01	.41	.03	.01	.01	.32	.03	.00	.01

Table 5: Congeneric Factor Structure

		1 Factor				3 Factor				5 Factor			
	N	α	ρ_+	λ_4	ω_t	α	ρ_+	λ_4	ω_t	α	ρ_+	λ_4	ω_t
Bias	50	-.01	.05	.01	.01	-.08	.08	.00	.02	-.07	.09	.00	.03
	100	-.01	.04	.01	.00	-.08	.06	.00	.00	-.07	.06	.00	.01
	400	.00	.02	.00	.00	-.07	.03	.00	.00	-.06	.03	.00	.00
	1000	.00	.01	.00	.00	-.07	.02	.00	.00	-.06	.02	.00	.00
	2000	.00	.01	.00	.00	-.07	.01	.00	.00	-.06	.01	.00	.00
Consistency	50	.03	.02	.03	.02	.05	.02	.04	.03	.05	.01	.03	.02
	100	.02	.02	.02	.02	.04	.02	.03	.03	.03	.01	.02	.02
	400	.01	.01	.01	.01	.02	.01	.01	.01	.01	.01	.01	.01
	1000	.01	.01	.01	.01	.01	.01	.01	.01	.01	.01	.01	.01
	2000	.00	.01	.01	.00	.01	.01	.01	.01	.01	.00	.00	.00
MSE * 100	50	.10	.33	.09	.06	.89	.71	.13	.13	.71	.78	.09	.13
	100	.05	.18	.05	.04	.76	.36	.09	.08	.53	.41	.04	.04
	400	.01	.05	.01	.01	.52	.10	.02	.02	.41	.11	.01	.01
	1000	.01	.02	.01	.01	.50	.04	.01	.01	.39	.04	.00	.00
	2000	.00	.01	.00	.00	.50	.02	.00	.00	.38	.02	.00	.00